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LETTER TO THE EDITOR

Complex recurrent space–times

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Abstract. Some similarities between 'P–P' wave space–times and complex recurrent and conformally symmetric space–times in general relativity are discussed.

A P–P wave in general relativity theory is a non-flat solution of Einstein's vacuum field equations which admits a covariantly constant null vector field l^a. The following three important characterizations of P–P waves are known (Ehlers and Kundt 1962). Firstly, P–P wave space–times are characterized as those non-flat vacuum space–times whose Riemann tensor R_{abcd} is *complex recurrent*,

$$\overset{+}{R}_{abcd;e} = \overset{+}{R}_{abcd}k_e, \qquad \overset{+}{R}_{abcd} = R_{abcd} + \mathrm{i}\overset{*}{R}_{abcd} \neq 0, \qquad R_{ab} = 0 \tag{1}$$

where R_{ab} is the Ricci tensor, k_a a (complex) vector field and an asterisk denotes the duality operator. (Latin indices take the values 0, 1, 2, 3, and a semi-colon denotes a covariant derivative.) Secondly, P–P wave space–times are characterized as those non-flat vacuum space–times which admit a (necessarily null) covariantly constant bivector. The principal null direction of this bivector is parallel to l^a. Finally, P–P wave space–times are characterized as those non-flat vacuum space–times whose infinitesimal holomony group is two-parameter perfect (Goldberg and Kerr 1961a, b). In this letter, space–times without the vacuum restriction will be considered, whose Weyl tensor C_{abcd} satisfies the generalized version of equation (1), namely

$$\overset{+}{C}_{abcd;e} = \overset{+}{C}_{abcd}k'_e, \qquad \overset{+}{C}_{abcd} = C_{abcd} + \mathrm{i}\overset{*}{C}_{abcd} \neq 0. \tag{2}$$

Such space–times are called *complex recurrent* if $k'_a \neq 0$ and *conformally symmetric* if $k'_a = 0$ (McLenaghan and Leroy 1972). The purpose of this letter is to point out certain similarities between the general space–times satisfying (2) and the P–P waves satisfying (1) by means of the above mentioned characterizations.

To obtain the first set of similarities between the space–times above, it is recalled that the Riemann tensor of a P–P wave is necessarily of Petrov type N with principal null direction parallel to l^a and that the *recurrence vector* k_a is necessarily a gradient vector, $k_{[a;b]} = 0$ (Ehlers and Kundt 1962). For the space–times satisfying (2), the Weyl tensor is either Petrov type N or D (McLenaghan and Leroy 1972).

For a type N space–time satisfying (2), one can establish the following results. Firstly, the Ricci scalar R' vanishes, and the principal null direction l'^a of C_{abcd} is easily shown to be *recurrent* ($l'_{a;b} = l'_a P_b$ for some vector field P_a). Secondly, the recurrence vector k'_a need not be a gradient vector but is a gradient if and only if a covariantly constant (necessarily null) bivector is admitted. The principal null direction of this bivector is parallel to l'^a.

Thirdly, the Ricci tensor can assume either of several different Segré types and its elementary divisors can be exhibited. In particular it assumes the Segré type $\{(2, 1, 1)\}$ with zero eigenvalue if and only if k_a is a gradient, the Ricci tensor taking the form $R_{ab} = A l_a l_b$ for some invariant A. Finally, of those type N space–times satisfying (2), those with k_a a gradient vector are characterized by their possessing a constant null vector field (necessarily parallel to l_a).

In the type D case, $R \neq 0$ and k_a is necessarily a (real) gradient $k_a = \frac{1}{6}R_{;a}$. A constant non-null bivector is always admitted and it is not difficult to show that the two principal null directions of the Weyl tensor and the two principal null directions of the constant non-null bivector coincide. The Ricci tensor turns out to have only simple elementary divisors and Segré type $\{(1, 1)(1, 1)\}$ with eigenvalues α and $\frac{1}{2}R - \alpha$ for some invariant α.

Next, it can be shown that if a space–time admits a constant non-zero null (non-null) bivector, the Weyl tensor is either zero or type N and satisfies (2) (either zero or type D and satisfies (2)). So a space–time satisfies (2) with k_a a gradient, possibly zero, if and only if it is not conformally flat and admits a constant non-zero bivector.

Finally, attention can be turned to the holonomy group structure of the fields satisfying (2). A classification of vacuum Riemann tensors based on holonomy group structure was first given by Schell (1961), and it was extended by Goldberg and Kerr (1961a, b) who demonstrated how Schell's classification separated off those (necessarily Petrov type N and III) non-flat vacuum space–times which admit recurrent null vector fields and who established the result mentioned in paragraph one. The essential ingredients of Schell's classification can be more easily derived by noting that since the left and right duals of a vacuum Riemann tensor are equal:

$$*R_{abcd} = \tfrac{1}{2}R^{mn}{}_{cd}\eta_{abmn} = \tfrac{1}{2}R_{ab}{}^{mn}\eta_{mncd} = R^{*}_{abcd} \tag{3}$$

it follows that if X_{ab} is a 'generating bivector' of R_{abcd}, (that is $X_{ab} = R_{abcd}Y^{cd}$ for some real bivector Y^{ab}) then $\overset{*}{X}_{ab} = \overset{*}{R}_{abcd}Y^{cd}$ and so $\overset{*}{X}_{ab}$ is also a generating bivector of R_{abcd} and equivalently $\overset{+}{X}_{ab} = \frac{1}{2}\overset{+}{R}_{abcd}Y^{cd}$ where $\overset{+}{X}_{ab} = X_{ab} + \mathrm{i}\overset{*}{X}_{ab}$. Similar remarks apply to the generating bivectors of the covariant derivatives of R_{abcd} and $\overset{+}{R}_{abcd}$. Hence the generating bivectors of a vacuum Riemann tensor and its covariant derivatives occur in 'dual' pairs and can be read off by consideration of the much simpler object $\overset{+}{R}_{abcd}$. In fact, *in vacuo*, $\overset{+}{R}_{abcd}$ can be expressed in terms of three complex self-dual bivectors (Sachs 1961) and this expression, together with the differential relations satisfied by these bivectors (Robinson and Schild 1963) easily leads to the holonomy group classification of the vacuum Riemann tensor. When the vacuum condition is dropped, the situation is not so simple. However it can be shown that for any space–time satisfying (2) with k_a a gradient vector field, the Riemann tensor can be expressed in terms of the real and imaginary parts of a self-dual complex recurrent bivector and so the holonomy group is in general two-parameter perfect. An interesting case when it is not occurs in the type D case. It can be shown that the Riemann tensor of a type D space–time satisfying (2) takes the form

$$R_{abcd} = (\alpha - \tfrac{1}{2}R)\overset{*}{S}_{ab}\overset{*}{S}_{cd} + \alpha S_{ab}S_{cd} \tag{4}$$

where α and $\frac{1}{2}R - \alpha$ are the Ricci eigenvalues mentioned earlier and S_{ab} is a covariantly constant non-null simple bivector spanned by the two principal null directions of the Weyl tensor. It then follows from (4) that if the holonomy group of R_{abcd} for such space–

times is not two-parameter perfect then it is one-parameter perfect and this latter condition occurs if and only if either $\alpha = 0$ or $\alpha = \frac{1}{2}R$. Thus the Ricci eigenvalues become 0 and $\frac{1}{2}R$. If the space–time was type D complex recurrent ($R \neq$ constant) then on putting $\alpha = 0$ or $\alpha = \frac{1}{2}R$ one finds that the space–time is *recurrent*;

$$R_{abcd;e} = R_{abcd}\omega_e, \qquad R_{abcd} \neq 0, \qquad \omega_a \neq 0 \tag{5}$$

where ω_a is a vector field and one is led to the space–times discussed by Thompson (1969). If the space–time was type D conformally symmetric, then the equation $R^b_{a;b} = 0$ reveals that α is a constant and the space–time is necessarily *symmetric* (cf McLenaghan and Leroy 1972)

$$R_{abcd;e} = 0, \qquad R_{abcd} \neq 0. \tag{6}$$

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